
An irreducible smooth non-admissible representation

by G.F. Helminck

Faculty of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, the Netherlands

Communicated by Prof. P.J. Zandbergen at the meeting of March 26, 1990**ABSTRACT**

It is shown for the group of k -rational points of an affine algebraic group G with k a finite extension of \mathbb{Q}_p that the topological irreducibility of unitary representations of G does not have to be equivalent to the algebraic irreducibility of the representation on the smooth vectors. We give for a specific G an example of an irreducible smooth representation, which is not admissible.

1.1. Let k be a finite extension of \mathbb{Q}_p . We denote by G the group of k -rational points of an affine algebraic group. It is a totally disconnected locally compact group. Let (ϱ, V) be a representation of G on the complex vector space V . A vector v in V is called smooth if the map

$$g \mapsto \varrho(g)(v), \quad g \in G,$$

is locally constant. The space of smooth vectors in V , V_∞ , is stable under the action of G and we denote this representation by ϱ_∞ . If $V = V_\infty$, then we call (ϱ, V) smooth. A smooth representation is called admissible if moreover the following condition holds: for each open subgroup K of G , the space of vectors $v \in V$ left fixed by $\varrho(K)$ is finite-dimensional.

We call a smooth representation irreducible if V and $\{0\}$ are the only G -modules in V and we call it pre-unitary if V carries a $\varrho(G)$ -invariant scalar product.

It was shown in [Ja] for reductive G and in [D] for unipotent G that every irreducible smooth representation (ϱ, V) of G is admissible. This allows you to show that a unitary representation (ϱ, V) of G is topologically irreducible if and

only if $(\varrho_\infty, V_\infty)$ is a smooth irreducible representation. We will show here that this does not hold for general G .

Take

$$G = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}, a \in k^*, x \in k \right\}.$$

It is the semi-direct product of

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a \in k^* \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in k \right\}.$$

Then the general theory of Mackey tells you that if you take a non-trivial character τ of N and induces this representation to one of G , one obtains a unitary irreducible representation $I(\tau)$ of G . We will identify τ with a character of k . The standard realization of $I(\tau)$ is on the space of measurable functions $f: G \rightarrow \mathbb{C}$ satisfying

$$(i) \quad f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \tau(x)f(g) \text{ a.e.}$$

$$(ii) \quad \int_{N \setminus G} f(g)\overline{f(g)}d\dot{g} < \infty,$$

with $d\dot{g}$ a right G -invariant measure on $N \setminus G$. G acts on this space by right translations. Clearly those functions are determined by their restriction to H and if we identify H with k^* , then we get a realization on $L^2(k^*)$ and the action of G is given by

$$(1.2) \quad \left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \cdot f \right)(b) = \tau(bx)f(ba).$$

For any totally disconnected locally compact group T , we denote the space of all locally constant functions with compact carrier by $C_c^\infty(T)$. Clearly $C_c^\infty(k^*)$ is contained in $L^2(k^*)_\infty$ and G -invariant. As we will see further on, it is also an irreducible smooth representation of G but it is not admissible: assume for simplicity that τ is nontrivial on \mathbb{O} the ring of integers of k , and trivial on $p = (\pi)$ the maximal ideal of \mathbb{O} . Denote for any subset A of k the characteristic function of A by χ_A . Consider the open subgroup K of G defined by

$$K = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{O}^*, x \in p \right\}.$$

From (1.2) and the assumption on τ , it follows that all the $\chi_{\pi^m \mathbb{O}^*}$, $m \geq 0$, in $C_c^\infty(k^*)$ are left fixed by K . It will then also be clear that if $\{\lambda_m\}$ is a sequence with $\lambda_m > 0$ for all m and $\sum_{m \geq 0} \lambda_m^2 < \infty$, then $f = \sum_{m \geq 0} \lambda_m \chi_{\pi^m \mathbb{O}^*}$ is left fixed by $\pi(K)$ and is an example of an element in $L^2(k^*)_\infty$, which does not belong to $C_c^\infty(k^*)$. In particular $(I(\tau)_\infty, L^2(k^*)_\infty)$ is not irreducible.

As for the irreducibility of $C_c^\infty(k^*)$, consider some non-zero f in $C_c^\infty(k^*)$ and let M be the span of its G -translates. To show that $M = C_c^\infty(k^*)$, it is sufficient to prove that all the χ_{1+p^m} , with m sufficiently large belong to M , for the

action of G includes all translations in k^* . The same argument allows you to assume that

$$f = \sum_{i \geq 0} f \chi_{\pi^i \mathbb{O}^*} = \sum_{i \geq 0} f_i, \text{ with } f_0 \neq 0.$$

Let g be in $C_c^\infty(k^*)$. Then we define for each unitary character σ of \mathbb{O}^* and each k in \mathbb{Z} , the function $g(\sigma, k)$ in $C_c^\infty(k^*)$ by

$$g(\sigma, k)(x) = g(x) \int_{\mathbb{O}^*} \tau(\pi^{1-k}bx) \sigma(b) d^*b.$$

Here d^*b denotes a Haar measure on \mathbb{O}^* . From the action of G it will be clear that if $g \in M$, then $g(\sigma, k)$ also belongs to M . Consider now the integral $G(\sigma, k)$ given by

$$G(\sigma, k) = \int_{\mathbb{O}^*} \tau(\pi^{1-k}b) \sigma(b) d^*b.$$

We list here some properties of $G(\sigma, k)$. First of all, if $\sigma \equiv 1$, then

$$\begin{aligned} G(1, k) &= \int_{\mathbb{O}} \tau(\pi^{1-k}b) db - \int_p \tau(\pi^{1-k}b) db \\ &= 0 \quad \text{if } k > 1 \\ &= -\text{vol}(p) \quad \text{if } k = 1 \\ &= \text{vol}(\mathbb{O}^*) \quad \text{if } k < 1. \end{aligned}$$

If σ is non-trivial, then there is a $n \geq 1$ such that $\sigma|1+p^n \equiv 1$ and $\sigma|1+p^{n-1} \not\equiv 1$. Then we have

$$\begin{aligned} G(\sigma, k) &= 0 \text{ if } k \neq n \\ |G(\sigma, n)|^2 &= (\text{vol}(\mathbb{O}^*) + \text{vol}(p)) \text{vol}(p^n). \end{aligned}$$

For if $k > n$

$$G(\sigma, k) = \sum_{b \in \mathbb{O}^*/1+p^n} \sigma(b) \int_{p^n} \tau(\pi^{1-k}b(1+t)) dt.$$

$$\parallel$$

$$0$$

If $n > k$

$$G(\sigma, k) = \sum_{b \in \mathbb{O}^*/1+p^{n-1}} \tau(\pi^{1-k}b) \int_{1+p^{n-1}} \sigma(bu) d^*u.$$

$$\parallel$$

$$0$$

If $n=k$, then the Gaussian sum $G(\sigma, k)$ satisfies

$$\begin{aligned}
 G(\sigma, n) \overline{G(\sigma, n)} &= \int_{\mathbb{O}^*} \int_{\mathbb{O}^*} \tau(\pi^{1-n}(b-c)) \sigma(bc^{-1}) d^*b d^*c \\
 &= \int_{\mathbb{O}^*} \int_{\mathbb{O}^*} \tau(\pi^{1-n}(u-1)c) \sigma(u) d^*u d^*c \\
 &= \int_{\mathbb{O}^*} \sigma(u) \left\{ \int_{\mathbb{O}} \tau(\pi^{1-n}(u-1)c) dc - \int_p \tau(\pi^{1-n}c(u-1)) dc \right\} d^*u \\
 &= \int_{1+p^n} \sigma(u) \cdot \text{vol}(\mathbb{O}^*) d^*u - \text{vol}(p) \int_{1+p^{n-1}/1+p^n} \sigma(u) d^*u \\
 &= (\text{vol}(\mathbb{O}^*) + \text{vol}(p)) \text{vol}(p^n).
 \end{aligned}$$

By considering a non-trivial character σ of $\mathbb{O}^*/1+p$ we see that the support of $f(\sigma, 1)$ is contained in \mathbb{O}^* and that $f(\sigma, 1) \neq 0$. Hence we may assume that f already had its support in \mathbb{O}^* . Note that if $x \in \mathbb{O}^*$ then

$$f(\sigma, k)(x) = f(x) \sigma(x)^{-1} \cdot G(\sigma, k).$$

Thus we have reduce the question to the following: if V is a nonzero subspace of $C^\infty(\mathbb{O}^*)$, which is stable under multiplication by unitary characters of \mathbb{O}^* and under translations in \mathbb{O}^* , is V then equal to $C^\infty(\mathbb{O}^*)$? The answer to this question is affirmative, since it boils down to the same question for the groups $\mathbb{O}^*/1+p^m$, $m > 0$.

REFERENCES

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